PERVERSE SHEAVES, KOSZUL IC-MODULES, AND THE QUIVER FOR THE CATEGORY $\mathcal{O}$

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ABSTRACT. For a stratified topological space we introduce the category of IC-modules, which are linear algebra devices with the relations described by the equation $d^2 = 0$. We prove that the category of (mixed) IC-modules is equivalent to the category of (mixed) perverse sheaves for flag varieties. As an application, we describe an algorithm calculating the quiver underlying the BGG category $\mathcal{O}$ for arbitrary simple Lie algebra, thus answering a question which goes back to I. M. Gelfand.

Dedicated to George Lusztig on the occasion of his 60-th birthday

1. INTRODUCTION

1.1. Motivated by R. MacPherson's cellular perverse sheaves, W. Soergel's studies of the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$, and L. Saper's L-modules we introduce here, in Section 2, an abelian category of IC-*modules* designed to be a derived-category-free model of the category of constructible perverse sheaves if the latter is Koszul (cf. 2.8.4 for the remarks beyond the Koszul case). In Section 3 we prove the following.

Theorem A. *The category of Schubert-constructible perverse sheaves on a flag variety is equivalent to the category of* IC-*modules.*

Moreover, we suggest a relationship between perverse sheaves and IC-modules for a much wider class of stratified spaces.

1.2. There exists a number of derived-category-free models of perverse sheaves. These include $\mathcal{D}$-modules, cf. e.g. [25] (which actually predate perverse sheaves), gluing constructions [1, 30], and "quiver" presentations, cf. e.g. [31, 19, 11, 9]. We believe that our "elementary" construction has the advantage of being well suited for *mixed* perverse sheaves. We construct a natural functor from mixed perverse sheaves to mixed IC-modules which is an equivalence in our two main examples: flag varieties and simplicial complexes, cf. Section 6.

1.3. Let G be a complex simple algebraic group, $\mathfrak{g} = \operatorname{Lie} G$ its Lie algebra, B a Borel subgroup, and $\mathfrak{b} = \operatorname{Lie} B$ its Lie algebra. Let $\mathfrak{h}$ be a Cartan of $\mathfrak{b}$, Φ the root system, and let W be the Weyl group. Let $\mathcal{O}_0$ be a regular block of the BGG category $\mathcal{O}$, cf. [5, 6, 7], equivalent to Schubert-stratification constructible perverse sheaves on G/B, [2, 3]. The simple objects of $\mathcal{O}_0$ are the simple $\mathfrak{g}$-modules with the highest weights $w(\rho) - \rho$ where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. If $\mathfrak{g} = \mathfrak{sl}_2$, then $\mathcal{O}_0$ is equivalent to the following category of linear algebra data

$$V_0 \underset{b}{\overset{a}{\rightleftarrows}} V_1$$

with the relation $b \circ a = 0$. This is the *quiver* of the category $\mathcal{O}_0$ for $\mathfrak{sl}_2$. The question of finding such a quiver for arbitrary simple Lie algebra $\mathfrak{g}$ goes back to I. M. Gelfand, cf. [18]. Seminal breakthroughs with implications for this question had been made in particular in [26, 2, 35, 4].

It is well known that the quiver has vertices labeled by the elements of the Weyl group W. If $y, w \in W$, then for every arrow from y to w there is an arrow from w to y. Moreover, the number of arrows from y to w is equal to $\mu(y, w)$ – the coefficient of the power $\frac{1}{2}(l(w) - l(y) - 1)$ of the Kazhdan-Lusztig polynomial [26]. However, the relations between arrows were not known explicitly and were difficult to obtain algorithmically, except for a few small rank cases, cf. [23, 24, 38].

In Section 5 we use IC-modules to encode the relations in the form $d^2 = 0$ and describe a *computer-executable algorithm* to calculate them explicitly. The algorithm is a refinement of an algorithm of C. Stroppel [38]. It works for arbitrary simple Lie algebra $\mathfrak{g}$, thus settling the Gelfand's question for the category $\mathcal{O}$. A detailed implementation of the algorithm for $\mathfrak{g} = \mathfrak{sl}_3$ is worked out in the Appendix (Section 8).

1.4. We believe that IC-modules should also be useful for the study of Harish-Chandra modules and Soergel's conjecture, cf. [37, Basic conjecture 1], as well as toric varieties, affine Grassmannians, and locally symmetric spaces, cf. Section 7.

Y. Drozd, S. Ovsienko, and V. Futorny who introduced me to Lie alge-
bras and quivers during my college years. The research was supported
in part by the NSF Postdoctoral Research Fellowship.

2. Definitions and The $d^2 = 0$ Theorem

2.1. **Setup.** Our sheaves are assumed to be sheaves of finitely gener-
ated R-modules over a ring R. Unless indicated otherwise, R is assumed
to be the field $\mathbb{C}$, but many results are true for other ground rings.

Let X be a topological space equipped with a sheaf of rings and let

$$X = \bigsqcup_{S \in \mathcal{S}} S$$

be a partition of X into a finite disjoint union of locally closed strata.
Let $\dim : \mathcal{S} \to \mathbb{Z}_{\geq 0}$ be a dimension function such that $\dim S \leq \dim S'$
if $\overline{S} \subseteq \overline{S}'$ and let us fix a perversity function $p : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$, cf. [22, 3].

Let L be a local system on S. Consider the IC-sheaf $\mathbf{IC}(\overline{S}, L) = \mathbf{IC}_p(\overline{S}, L)$, cf. [21, 22, 3] corresponding to L and the perversity p on
the closure $\overline{S}$ of the stratum S, and let

$$IH^*(\overline{S}, L) = IH_p^*(\overline{S}, L) = \mathbb{H}^*(\mathbf{IC}_p(\overline{S}, L))$$

be the hypercohomology of the sheaf $\mathbf{IC}_p(\overline{S}, L)$. From now on we will
omit perversity from the notation. Perversity is always assumed to be
the middle one for algebraic varieties with algebraic stratifications. No-
tice that $IH^*(\overline{S}, L)$ is naturally a graded $H^*(X)$-module, where $H^*(X)$
is the cohomology of X with coefficients in R.

From now on we assume that all strata are *contractible* unless ex-
plicitly specified otherwise, e.g. in section 2.5.

2.2. **Definition.** A *mixed* IC-module V is the following data:

(1) for every stratum $S \in \mathcal{S}$, a $\mathbb{Z}$-graded R-vector space (stalk)
$\oplus_{i \in \mathbb{Z}} V_S^i$ on S,
(2) for every pair of incident (i.e. one is in the closure of the other)
strata S and S' such that $S \neq S'$, a boundary map

$$v(S, S') \in \mathrm{Hom}_{H^*(X)}^{(1,1)}(IH^*(\overline{S}, V_S^\bullet), IH^*(\overline{S}', V_{S'}^\bullet)),$$

where $\mathrm{Hom}^{(1,1)}$ is the space of degree $(1,1)$ morphisms of $H^*(X)$-
modules $IH^*(\overline{S}, V_S^\bullet)$ bigraded by the degree of IH^* and the degree
of $V_S^\bullet$. The data is subject to the

(2.2.1)   Chain Complex Axiom: $d^2 = 0$,

where d is the degree $(1,1)$ map

$$(2.2.2) \qquad d = \bigoplus_{S,S'} v(S,S') : \oplus_S IH^*(\overline{S}, V_S^\bullet) \to \oplus_S IH^*(\overline{S}, V_S^\bullet).$$

The bigraded space $\oplus_S IH^*(\overline{S}, V_S^\bullet)$ with the differential d is called the *total complex* of the IC-module V. Its cohomology is naturally bigraded.

2.3. A *non-mixed* IC module is defined exactly as above, except the stalks V_S are just (non-graded) R-vector spaces, and boundary maps

$$v(S,S') \in \mathrm{Hom}^1_{H^*(X)}(IH^*(\overline{S}), IH^*(\overline{S}')) \otimes \mathrm{Hom}_R(V_S, V_{S'})$$

and the differential d are maps of degree 1. The cohomology of the total complex has only one natural grading.

2.4. A morphism of IC modules is the collection of maps between stalks commuting with the boundary maps. The abelian category of non-mixed IC-modules of finite length (i.e. having a finite filtration with simple subquotients) will be denoted by $\mathcal{A}_\mathcal{S}(X)$.

The category of mixed IC-modules of finite length will be denoted by $\mathcal{A}_\mathcal{S}^{\mathrm{mixed}}(X)$. It is a mixed category in the sense of [4, 4.1]. A mixed IC-module V is called *pure* of weight m if it is concentrated in degree $-m$ i.e., $V_S^i = 0$ unless $i = -m$ for all $S \in \mathcal{S}$. We have the obvious functor $v : \mathcal{A}_\mathcal{S}^{\mathrm{mixed}}(X) \to \mathcal{A}_\mathcal{S}(X)$ forgetting the mixed structure.

2.5. **Local Systems.** One could further generalize the definition of an IC-module to the case of non-simply-connected strata. In general, the stalk V_S is a local system on S, and the boundary maps are

$$v(S,S') \in \mathrm{Hom}^1_{H^*(X)}(IH^*(\overline{S}, V_S), IH^*(\overline{S}', V_{S'})),$$

where Hom^1 are the degree 1 morphisms of $H^*(X)$-modules. (We consider here the non-mixed case to simplify the notation.) The Chain Complex Axiom and the total complex are precisely as above.

From now on we will consider non-mixed IC-modules unless explicitly specified otherwise, e.g. in Section 6.

2.6. **Verdier duality.** We will only consider here the middle perversity case in order to simplify the notation. For an IC-module V the stalks of its Verdier dual IC-module DV are defined as follows:

$$DV_S = V_S^*,$$

where V_S^* is the dual vector space. Notice that

$$(V_S \otimes IH^i(\overline{S}))^* = V_S^* \otimes IH^{-i}(\overline{S})$$

since according to [22, 5.3] we have $IH^i(\overline{S}) = (IH^{-i}(\overline{S}))^*$. Now the boundary maps

$$dv^i(S, S') : \quad \begin{array}{ccc} V_S^* \otimes IH^i(\overline{S}) & \to & V_{S'}^* \otimes IH^{i+1}(\overline{S}') \\ || & & || \\ V_S^* \otimes (IH^{-i}(\overline{S}))^* & \to & V_{S'}^* \otimes (IH^{-i-1}(\overline{S}'))^* \end{array}$$

of DV are $dv^i(S, S') = (v^{-i-1}(S', S))^*$, where

$$v^{-i-1}(S', S) : V_{S'} \otimes IH^{-i-1}(\overline{S}') \to V_S \otimes IH^{-i}(\overline{S})$$

are the boundary maps of V. The Chain Complex Axiom (2.2.1) for DV is obviously satisfied. We have constructed a contravariant functor $D : \mathcal{A}_S(X) \to \mathcal{A}_S(X)$. Clearly, $D \circ D = \text{Id}$.

2.7. Denote the category of $\mathcal{S}$-constructible perverse sheaves of finite length on X by $\mathcal{P}_S(X)$, cf. [3]. We present here a more precise formulation of Theorem A. The *raison d'être* of these notes is the following.

The $d^2 = 0$ Theorem. *If X is a flag variety stratified by Schubert cells, or a simplicial complex stratified by simplices, then*

(i) *the categories*

$$\mathcal{P}_S(X) \simeq \mathcal{A}_S(X)$$

are equivalent,

(ii) *the total complex of an* IC*-module calculates the hypercohomology of the corresponding perverse sheaf.*

Remark: The $d^2 = 0$ Thereom for simplicial complexes. For perverse sheaves on simplicial complexes constructible with respect to the triangulation we have

$$IH^*(\overline{\Delta}) = \mathbb{C}[\delta(\dim \Delta)],$$

where Δ is a simplex and δ is a cellular perversity, [29, 33, 40]. The IC-modules in this case are *simplicial perverse sheaves* of R. MacPherson and the statement of the The $d^2 = 0$ Thereom is a theorem due to MacPherson [29], cf. [33, 39, 40] for an alternative proof.

2.8. **Restrictions, relaxing remarks, and conjectures.** We use only a limited number of properties (some are listed below) of the category of perverse sheaves in order to prove the Theorem. It is natural to conjecture that under these restrictions The $d^2 = 0$ Theorem holds. (Note that only the statement (ii) is in question.)

2.8.1. Assume that we have only finitely many strata $S \in \mathcal{S}$. This assumption can be relaxed, but we will not pursue it here.

2.8.2.   For the purposes of this text we will assume that the strata
are contractible. This assumption could be relaxed: The $d^2 = 0$ The-
orem with a local system version of IC-modules is a tautology for a
stratification with one stratum, for example.

2.8.3.   Assume that the category $\mathcal{P}_\mathcal{S}(X)$ has enough projectives i.e.,
every object has a projective resolution.

2.8.4.   We assume that the category $\mathcal{P}_\mathcal{S}(X)$ is Koszul i.e., it is equiva-
lent to a category of finitely generated modules over a Koszul algebra.
When the category $\mathcal{P}_\mathcal{S}(X)$ is not Koszul one can still reconstruct it
from the algebra $\mathrm{Ext}^*_{\mathcal{P}_\mathcal{S}(X)}(L, L)$, where L is the direct product of all
simple objects in $\mathcal{P}_\mathcal{S}(X)$ with the canonical A_∞-structure, cf. e.g. [28,
Problem 2]. One has to take this A_∞-structure into account in order
to generalize the IC-modules beyond the Koszul case.

2.8.5.   For any two incident strata S and S' and two simple objects
$\mathcal{L}_1 = \mathbf{IC}(S, L_S)$ and $\mathcal{L}_2 = \mathbf{IC}(S', L_{S'})$ in $\mathcal{P}_\mathcal{S}(X)$ we assume that we
have the isomorphisms

$$\mathrm{Ext}^i_{\mathcal{P}_\mathcal{S}(X)}(\mathcal{L}_1, \mathcal{L}_2) = \mathrm{Ext}^i_{D(X)}(\mathcal{L}_1, \mathcal{L}_2) = \mathrm{Hom}^i_{H^*(X)}(\mathbb{H}^*(\mathcal{L}_1), \mathbb{H}^*(\mathcal{L}_2)),$$

and $\mathrm{Ext}^i_{\mathcal{P}_\mathcal{S}(X)}(\mathcal{L}_1, \mathcal{L}_2) = \mathrm{Ext}^i_{D(X)}(\mathcal{L}_1, \mathcal{L}_2) = 0$ for any two non-incident
strata S and S'. Here Hom^i is the space of $H^*(X)$-morphisms of degree
i. In particular, the representation of the algebra $\mathrm{Ext}^*_{\mathcal{P}_\mathcal{S}(X)}(L, L)$ in the
space $\oplus_S IH^*(\overline{S}, L_S)$ is faithful. One could generalize IC-modules using
another faithful representation of the algebra $\mathrm{Ext}^*_{\mathcal{P}_\mathcal{S}(X)}(L, L)$.

3. Proof of Theorem A

In this section we prove The $d^2 = 0$ Theorem, and thus Theorem A,
following a suggestion of W. Soergel. In fact our proof works for any
stratified space with the restrictions 2.8 and one additional condition:
$R_X[\dim X]$ is a simple object of $\mathcal{P}_\mathcal{S}(X)$.

3.1.   Let G be a simple algebraic group over $\mathbb{C}$, $B \subseteq P \subseteq G$ be a Borel
subgroup and a parabolic subgroup, W_P the subgroup of the Weyl
group W and let $X = G/P$ be the corresponding partial flag variety,
stratified by Schubert cells X_w, where w is a coset in W/W_P which
we identify with the set of strata $W_\mathcal{S}$ of X. Denote by $\mathcal{P}_{\mathrm{Schubert}}(G/P)$
the category of Schubert-stratification constructible perverse sheaves
on G/P.

3.2. **Theorem.** [35, 20] *For any $y, w \in W_{\mathcal{S}}$ the natural map*

$$\mathrm{Ext}^*_{D(X)}(\mathbf{IC}_y, \mathbf{IC}_w) = \mathrm{Hom}^*_{H^*(X)}(IH^*(\overline{X}_y), IH^*(\overline{X}_w))$$

is an isomorphism of graded spaces. (Here Hom^i denotes degree i morphisms of $H^*(X)$-modules.) Moreover, the left and right hand sides are isomorphic as graded algebras.

3.3. **Koszul complex.** In this subsection $A = \oplus_{i \geq 0} A_i$ is an arbitrary Koszul algebra with semisimple $A_0 = k$ over the field $\mathbb{C}$. All tensor products till the end of Section 3 are over k unless specified otherwise. Let $A^!$ be the quadratic dual algebra and let us consider the Koszul complex [4]

$$(3.3.1) \qquad \cdots A \otimes {}^*(A_2^!) \to A \otimes {}^*(A_1^!) \to A \to k.$$

Recall the construction of the differential from [4, 2.7-2.8]. Identify $A \otimes {}^*(A_i^!) = \mathrm{Hom}_{-k}(A_i^!, A)$. Let $\mathrm{Id}_{A_1} = \sum a'_\alpha \otimes a_\alpha \in A_1^! \otimes A_1$ under the canonical isomorphism $\mathrm{Hom}_k(A_1, A_1) = A_1^! \otimes A_1$. Observe that a_α (resp. a'_α) is a basis in A_1 (resp. $A_1^!$). Then the differential

$$d : A \otimes {}^*(A_{i+1}^!) \to A \otimes {}^*(A_i^!)$$

is constructed as follows: $(df)(a) = \sum f(aa'_\alpha)a_\alpha$ for $f \in \mathrm{Hom}_{-k}(A_{i+1}^!, A)$, $a \in A_i^!$.

Now, the entries of the Koszul complex are A-k-bimodules. Let $k = \prod_y \mathbb{C}_y$ be a product of fields $\mathbb{C}$ and let y be the identities in these fields (local idempotents). Let us fix $e \in k$ to be such a local idempotent. We can multiply the Koszul complex by e from the right.

$$(3.3.2) \qquad \cdots A \otimes {}^*(A_2^!)e \to A \otimes {}^*(A_1^!)e \to Ae \to \mathbb{C}_e,$$

where $\mathbb{C}_e$ is the field corresponding to e considered as a simple left A-module. This is a projective resolution of $\mathbb{C}_e$.

Let M be a finitely generated left module over A. Applying $\mathrm{Hom}_A(\_, M)$ to the complex (3.3.2) we get the complex

$$(3.3.3) \qquad \cdots e(A_2^!) \otimes M \leftarrow e(A_1^!) \otimes M \leftarrow eM$$

calculating $\mathrm{Ext}^*_A(\mathbb{C}_e, M)$ with the differential $d : e(A_i^!) \otimes M \to e(A_{i+1}^!) \otimes M$ given by $d(ea \otimes m) = \sum eaa'_\alpha \otimes a_\alpha m$ for $a \in A_i^!$ and $m \in M$.

For a Koszul ring we have an algebra isomorphism [4, Theorem 2.10.1]

$$\mathrm{Ext}^*_A(k, k) = (A^!)^{\mathrm{opp}}$$

which induces an isomorphism of k-k-bimodules (with switched left and right action)

$$\mathrm{Ext}^i_A(k, k) = A_i^!,$$

and the isomorphism

$$\oplus_y \operatorname{Ext}^i_A(\mathbb{C}_e, \mathbb{C}_y) = \operatorname{Ext}^i_A(\mathbb{C}_e, k) = e A^!_i,$$

where $k = \prod_y \mathbb{C}_y$.

Rewriting the complex (3.3.3) we get

$$(3.3.4) \quad \cdots \oplus_y (\operatorname{Ext}^2_A(\mathbb{C}_e, \mathbb{C}_y) \otimes yM) \leftarrow \oplus_y (\operatorname{Ext}^1_A(\mathbb{C}_e, \mathbb{C}_y) \otimes yM) \leftarrow eM.$$

3.4. **Flag varieties.** Now let us apply this general machinery to our situation. Recall that perverse sheaves on $X = G/P$ are modules over a Koszul algebra A and $k = \prod_{W_\mathcal{S}} \mathbb{C}_y$ where $W_\mathcal{S}$ is the set of Schubert strata, cf. [4]. Let e be the idempotent corresponding to the open Schubert stratum. Then, as a simple perverse sheaf $\mathbb{C}_e = \mathbb{C}_X[\dim X]$ is the shifted constant sheaf on the variety X.

Let M be a finitely generated left A-module and let us denote by the same letter the corresponding perverse sheaf on X. Let $\mathcal{D}(X, W_\mathcal{S})$ denote the derived category of sheaves smooth along the Schubert stratification. We have

$$(3.4.1)$$
$$\operatorname{Ext}^i_A(\mathbb{C}_e, M) = \operatorname{Hom}_{\mathcal{D}(X, W_\mathcal{S})}(\mathbb{C}_X[\dim X], M[i]) = \mathbb{H}^{i - \dim X}(M).$$

Here the first equality is due to [4, Corollary 3.3.2] and the second is by definition of hypercohomology. In particular,

$$\operatorname{Ext}^i_A(\mathbb{C}_e, \mathbb{C}_y) = \mathbb{H}^{i - \dim X}(\mathbf{IC}(\overline{X}_y)) = IH^{i - \dim X}(\overline{X}_y),$$

where X_y is the Schubert cell corresponding to the idempotent $y \in k$.

Thus we can rewrite the complex (3.3.4) as follows

$$(3.4.2) \quad \cdots \bigoplus_y (IH^{1 - \dim X}(\overline{X}_y) \otimes_k yM) \leftarrow \bigoplus_y (IH^{0 - \dim X}(\overline{X}_y) \otimes_k yM)$$

with the differential

$$d : \bigoplus_y (IH^i(\overline{X}_y) \otimes_k yM) \to \bigoplus_y (IH^{i+1}(\overline{X}_y) \otimes_k yM)$$

obtained as follows:

$$(3.4.3) \qquad\qquad d(c_y \otimes ym) = \left(\sum a'_\alpha \cdot c_y \otimes a_\alpha ym \right)$$

for $c_y \in IH^i(\overline{X}_y)$, $m \in M$ and $a'_\alpha \cdot c_y$ is the *left* action of $\operatorname{Ext}^*_A(k, k)$ (identified with the *right* action of $A^!$) on $\oplus_y IH^i(\overline{X}_y)$. Let us also write down the formula for d^2:

$$(3.4.4) \qquad\qquad d^2(c \otimes m) = \sum_{\alpha, \beta} (a'_\beta a'_\alpha \cdot c \otimes a_\beta a_\alpha \cdot m)$$

3.5. **Proposition.** The action of the tensor algebra $T(A_1) = k \oplus \bigoplus_{i=1}^{\infty} A_1^{\otimes i}$ on its finitely generated left module M descends to the action of A on M if and only if the sequence of maps (3.4.2) is a complex i.e., $d^2 = 0$.

First we prove the following two lemmas.

3.6. **Lemma.** Let M be a finitely generated left $T(A_1)$-module. We have $d^2 = 0$ for d constructed as in (3.4.2) if and only if $(d')^2 = 0$ in the sequence of maps

$$(3.6.1) \qquad \mathrm{Ext}_A^2(k, k) \otimes M \xleftarrow{d'} \mathrm{Ext}_A^1(k, k) \otimes M \xleftarrow{d'} M,$$

where

$$d'(m) = \sum_{\alpha} a'_{\alpha} \otimes a_{\alpha} m \qquad \text{and} \qquad d'(g \otimes m) = \sum_{\alpha} a'_{\alpha} \cdot g \otimes a_{\alpha} m$$

for $g \in \mathrm{Ext}_A^1(k, k)$ and $m \in M$.

Proof. Indeed,

$$(3.6.2) \qquad (d')^2(m) = \sum_{\alpha, \beta} a'_{\beta} a'_{\alpha} \otimes a_{\beta} a_{\alpha} \cdot m \in \mathrm{Ext}_A^2(k, k) \otimes M$$

Consider a tensor product of the isomorphism of the Theorem 3.2 and the identity:

$$\phi : \mathrm{Ext}_A^2(k, k) \otimes_k M \to \mathrm{Hom}_{H^*(X)}^2(\oplus_y IH^*(\overline{X}_y), \oplus_y IH^*(\overline{X}_y)) \otimes_k M.$$

For $c \in \oplus_y IH^*(\overline{X}_y)$ let

$$\mathrm{ev}_c : \mathrm{Hom}_{H^*(X)}^2(\oplus_y IH^*(\overline{X}_y), \oplus_y IH^*(\overline{X}_y)) \otimes_k M \to \oplus_y IH^*(\overline{X}_y) \otimes M$$

be the map evaluating the first tensor multiple (i.e. Hom^2) at c. Comparing the formulas (3.4.4) and (3.6.2) we see that

$$\mathrm{ev}_c(\phi((d')^2(m))) = d^2(c \otimes m),$$

where $c \in \oplus_y IH^*(\overline{X}_y)$, $m \in M$, and d^2 is constructed as in (3.4.4). Since ϕ is an isomorphism, the statement of the lemma follows. $\qquad \square$

Recall that $A = T(A_1)/\langle R \rangle$ where $\langle R \rangle$ is the ideal of relations generated by $R \subset (A_1 \otimes_k A_1)$.

3.7. **Lemma.** The map $(d')^2$ defined in (3.6.1) is equal to zero for a finitely generated left $T(A_1)$-module M if and only if $r \cdot M = 0$ for all $r \in R \subseteq (A_1 \otimes A_1)$.

Proof. Let us rewrite the complex (3.6.1) as

$$(3.7.1) \qquad R^* \otimes M \xleftarrow{d'} A_1^! \otimes M \xleftarrow{d'} M,$$

where $R^* = (A_1 \otimes A_1)^*/R^\perp = A_2^!$. We can give another description of the map $(d')^2$ as follows. Denote $V = A_1 \otimes A_1$, and $V^* = (A_1 \otimes A_1)^*$. Consider the map

$$\mathrm{Id}_{V^*} \otimes \mathrm{act} : V^* \otimes V \otimes M \to V^* \otimes M,$$

where act is the action of $V = A_1 \otimes A_1$ on M, and the tensor product of the quotient map $V^* \to V^*/R^\perp$ and Id_M:

$$p : V^* \otimes M \to V^*/R^\perp \otimes M = A_2^! \otimes M.$$

Let $\psi = p \circ (\mathrm{Id}_{V^*} \otimes \mathrm{act})$. Then for $m \in M$

$$(d')^2(m) = \psi(\sum_{\alpha,\beta}(a'_\alpha \otimes a'_\beta) \otimes (a_\beta \otimes a_\alpha) \otimes m).$$

(Remember that *left* multiplication in $\mathrm{Ext}_A^*(k, k)$ is identified with the *right* multiplication in $A^!$.) Taking advantage of the identities $V^* \otimes V = \mathrm{Hom}_k(V, V)$, $V^* \otimes M = \mathrm{Hom}_k(V, M)$, and $A_2^! \otimes M = \mathrm{Hom}_k(R, M)$ we can consider ψ as the composite map

$$\mathrm{Hom}_k(V, V) \otimes M \to \mathrm{Hom}_k(V, M) \to \mathrm{Hom}_k(R, M).$$

Now for $m \in M$

$$(d')^2(m) = \psi(\mathrm{Id}_V \otimes m) \in \mathrm{Hom}_k(R, M)$$

and $(d')^2(m)(r) = r \cdot m$ for $r \in R$. Thus $(d')^2 = 0$ if and only if $r \cdot m = 0$ for any $r \in R$ and any $m \in M$. $\qquad \square$

Proof of Proposition 3.5. If the action of $T(A_1)$ descends to the action of A, that is if M is an A-module then (3.4.2) is a complex by construction.

Suppose we are given a $T(A_1)$-module M and suppose that (3.4.2) is a complex. Then by Lemma 3.6 and Lemma 3.7 we have $r \cdot m = 0$ for any $r \in R$ and any $m \in M$. $\qquad \square$

3.8. **From $T(A_1)$-modules to IC-modules.** Notice that a finitely generated left (non-graded) $T(A_1)$-module M for which the sequence of maps (3.4.2) is a complex is exactly the same thing as a (non-mixed) IC-module V of finite length defined in 2.2 and 2.3. Indeed:

3.8.1. The stalks $V_y = yM$ for all $y \in W_{\mathcal{S}}$. (Here we abuse notation by denoting a stratum of $X = G/P$ and the corresponding local idempotent by the same letter y.)

3.8.2. The boundary maps $v(y, w)$, or rather their sum $d = \oplus_{y,w} v(y, w)$ is given by

$$
\bigoplus_y IH^*(X_y) \otimes yM \simeq \bigoplus_y IH^*(X_y) \otimes k \otimes yM
$$

$$
\xrightarrow{\gamma_1} \bigoplus_y IH^*(X_y) \otimes (A_1^! \otimes A_1) \otimes yM
$$

$$
\xrightarrow{\gamma_2} \bigoplus_y IH^*(X_y) \otimes yM
$$

where γ_1 is the canonical homomorphism $k \to A_1^! \otimes A_1$ tensored with $\mathrm{Id}'s$; and γ_2 is the tensor product of the *right* action of $A_1^!$ on $\oplus_y IH^*(X_y)$ (which is identified with the *left* action of $\mathrm{Ext}_A^1(k, k)$ on $\oplus_y IH^*(X_y)$ cf. (3.4.3)) and the action of A_1 on $\oplus_y yM = M$.

3.8.3. The Chain Complex Axiom is clearly equivalent to the requirement that (3.4.2) is a complex. In fact, (3.4.2) is the total complex of the IC-module.

3.9. **Last steps of the proof of The $d^2 = 0$ Theorem.** The complex (3.4.2) calculates the hypercohomology of M considered as a perverse sheaf due to the fact that the complex (3.3.2) is a projective resolution of $\mathbb{C}_e$ and the formula 3.4.1. The $d^2 = 0$ Theorem is proved.

3.10. **Example:** $\mathbb{P}^1$ [32]. Let $X = \mathbb{P}^1$ stratified by two strata $X = S_0 \sqcup S_1$ where $S_0 = \mathbb{C}^0 = \mathrm{pt}$, and $S_1 = \mathbb{C}^1$. We have

$$
IH^*(\overline{S}_0) = IH^*(\mathrm{pt}) = \mathbb{C}[0],
$$

where $\mathbb{C}[0]$ is the complex with zero differential and one-dimensional space in position 0, and

$$
IH^*(\overline{S}_1) = IH^*(\mathbb{P}^1) = \mathbb{C}[1] \oplus \mathbb{C}[-1],
$$

where $\mathbb{C}[1] \oplus \mathbb{C}[-1]$ is the complex with zero differential and two one-dimensional spaces in positions -1 and 1. Now the spaces

$$
\mathrm{Hom}^1_{H^*(\mathbb{P}^1)}(IH^*(\mathbb{P}^1), IH^*(\mathrm{pt})) \quad \text{and} \quad \mathrm{Hom}^1_{H^*(\mathbb{P}^1)}(IH^*(\mathrm{pt}), IH^*(\mathbb{P}^1))
$$

are one dimensional. We will schematically depict a non-zero element
of the these spaces as

$$\mathbb{C}[0] \qquad\qquad\qquad\qquad \mathbb{C}[0]$$

$$\nearrow \qquad\qquad \text{and} \qquad\qquad \searrow$$

$$\mathbb{C}[1] \qquad \mathbb{C}[-1] \qquad\qquad \mathbb{C}[1] \qquad \mathbb{C}[-1]$$

respectively. Now we see that an IC-module may be depicted as the
following diagram

$$V_0$$

$$a \nearrow \quad \searrow b$$

$$V_1 \qquad\qquad V_1$$

$$\text{degree}: \quad -1 \quad\quad 0 \quad\quad 1$$

with the relation $d^2 = 0$ i.e., $b \circ a = 0$. Thus the category of IC-modules
in this case is isomorphic to the category of quiver representations

$$V_0 \underset{b}{\overset{a}{\rightleftarrows}} V_1$$

with the relation $b \circ a = 0$.

4. Some combinatorics of Schubert varieties

We start with some well known combinatorial preliminaries. In this
section $X = G/B$ is a full flag variety associated to a simple algebraic
group G over $\mathbb{C}$.

4.1. **Cohomology ring.** Let us denote $C = H^*(X)$. Following [36],
let $S(\mathfrak{h})$ be the symmetric algebra over the Cartan $\mathfrak{h}$ of $\mathfrak{g} = \operatorname{Lie} G$, let
$S_+(\mathfrak{h}) \subset S(\mathfrak{h})$ be the elements of positive degree and let $S_+(\mathfrak{h})^W$ be the
ideal generated by the W-invariants of $S_+(\mathfrak{h})$. (Here the invariants are
with respect to the usual "linear" action, cf. [36].) Recall that

$$C = H^*(G/B) = S(\mathfrak{h})/S_+(\mathfrak{h})^W.$$

Let B^- be the opposite Borel. Denote by

$$Y_w = B^- wB/B$$

the opposite Schubert variety associated to $w \in W$, and denote by σ_w
the class $[Y_w]$ in $H^{2l(w)}(X)$. It is well known that the ring $H^*(X)$ is
generated by the classes

$$\sigma_{s_i}, \qquad i = 1, \ldots, r,$$

where s_i are reflections with respect to simple roots and r is the rank of $\mathfrak{g} = \operatorname{Lie} G$. Moreover, we have the following Chevalley's formula, cf. [14], for multiplying the Schubert classes in $C = H^*(X)$

$$(4.1.1) \qquad \sigma_{s_i} \cdot \sigma_w = \sum_{\substack{\alpha \in \Phi^+ \\ l(ws_\alpha)=l(w)+1}} \langle \breve{\omega}_i, \alpha \rangle \frac{(\alpha_i, \alpha_i)}{(\alpha, \alpha)} \sigma_{ws_\alpha},$$

where $\breve{\omega}_i$ is the i-th fundamental coweight, and α_i is the i-th simple root, and $\langle \breve{\omega}_i, \alpha_j \rangle = \delta_{ij}$.

4.2. Let s_i, $i \in \{1, \ldots, r\}$ be a simple reflection in W and let C^{s_i} be the algebra of invariants of s_i. It is a subalgebra of C. We have the following description, explained to us by A. Postnikov, cf. [14]:

$$C^{s_i} = \operatorname{span}\{\sigma_w | ws_i > w, \ w \in W\}.$$

Moreover, for any s_i, $i \in \{1, \ldots, r\}$ we have

$$(4.2.1) \qquad\qquad C = \sigma_{s_i} \cdot C^{s_i} \oplus C^{s_i}$$

4.3. Now let us denote

$$\mathbb{V}_w = IH^*(\overline{X}_w).$$

It is a graded C-module. Due to Soergel, we have the following inductive description of $\mathbb{V}_w$. Take a reduced decomposition $w = s_1 \ldots s_l$ Denote

$$C_{s_1 \ldots s_l} = C \otimes_{C^{s_l}} \otimes \ldots C \otimes_{C^{s_1}} \mathbb{C}.$$

The action of C of the leftmost tensor factor C equips $C_{s_1 \ldots s_l}$ with a C-module structure. By [35, 36] we know that

$$(4.3.1) \qquad\qquad C_{s_1 \ldots s_l} = \mathbb{V}_w \oplus \bigoplus_{y<w} \mathbb{V}_y^{n(y)}$$

as a C-module.

Example. Type A_2.
(1) $w = s_1 s_2$: $C_{s_1 s_2} = \mathbb{V}_w$,
(2) $w = s_1 s_2 s_1$: $C_{s_1 s_2 s_1} = \mathbb{V}_w \oplus \mathbb{V}_{s_1}$, and $C_{s_2 s_1 s_2} = \mathbb{V}_w \oplus \mathbb{V}_{s_2}$.

4.4. The C-module $C_{s_1 \ldots s_l}$ has a basis

(4.4.1) $$a_l \otimes \cdots \otimes a_1 \otimes 1,$$

where each

$$a_i = \begin{cases} \text{either} & 1, \\ \text{or} & \sigma_{s_i}. \end{cases}$$

In particular, $\dim C_{s_1 \ldots s_l} = 2^l$. This basis is homogeneous:

$$\deg a_l \otimes \cdots \otimes a_1 \otimes 1 = \deg a_l + \cdots + \deg a_1,$$

where $\deg 1 = -1$ and $\deg \sigma_{s_i} = 1$. We know the action of

$$\sigma_{s_i}, \ i = \{1, \ldots, r\}$$

on this basis by induction, the decomposition 4.2.1, and the Chevalley's formula (4.1.1).

4.5. We can now give a description of IC-modules purely in terms of combinatorics of the ring $C = H^*(X)$. Recall that for each $w \in W$ we have a *graded* C-module $\mathbb{V}_w = IH^*(\overline{X}_w)$ whose combinatorial description is given in 4.3. Now a stalk of an IC-module M on $X = G/B$ is a vector space M_w for every $w \in W$, and for every pair of $y, w \in W$ we have boundary maps

$$m(y, w) \in \mathrm{Hom}_C^1(\mathbb{V}_y, \mathbb{V}_w) \otimes \mathrm{Hom}(M_y, M_w)$$

subject to the usual Chain Complex Axiom (2.2.1). Let us denote this category of IC-modules by $\mathcal{A}_{\mathrm{Schubert}}(G/B)$.

4.6. **Localization.** By the Beilinson-Bernstein Localization Theorem (used in [2] to prove the Kazhdan-Lusztig conjecture [26], cf. [27, 13]) and The $d^2 = 0$ Theorem we obtain

Theorem. *The categories*

$$\mathcal{O}_0 \simeq \mathcal{P}_{\mathrm{Schubert}}(G/B) \simeq \mathcal{A}_{\mathrm{Schubert}}(G/B)$$

are equivalent.

5. Quiver algorithm

Let us keep the setup of the previous section. Using Theorem 4.6 we can calculate the relations between the arrows of the quiver for the category $\mathcal{O}_0$ in the following way.

5.1. For $y, w \in W$ we have exactly $\mu(y, w)$ (note that $\mu(y, w)$ could be 0) arrows from y to w. Let us denote them by $a_{y,w}^1, \ldots, a_{y,w}^{\mu(y,w)}$, and consider these arrows as a basis in A_1, cf. Section 3.3. Suppose we have a basis in the vector space

$$\operatorname{Hom}_{H^*(X)}^1(IH^*(\overline{X}_y), IH^*(\overline{X}_w)) = \operatorname{Hom}_C^1(\mathbb{V}_y, \mathbb{V}_w)$$

given by matrices

(5.1.1) $$A_{y,w}^1, \ldots, A_{y,w}^{\mu(y,w)} \in \operatorname{Hom}_C^1(\mathbb{V}_y, \mathbb{V}_w)$$

with respect to some homogeneous bases in $\mathbb{V}_y$ and $\mathbb{V}_w$. Consider a matrix with elements in $T(A_1)$, cf. 3.5

$$\widetilde{d}_{y,w} = \sum_{i \in \{1, \ldots, \mu(y,w)\}} a_{y,w}^i A_{y,w}^i.$$

Denote $\widetilde{d} = \sum_{y,w} \widetilde{d}_{y,w}$. Then all the relations between the arrows are encoded by the equation $\widetilde{d}^2 = 0$, or more precisely the linear relations between paths of length 2 going from y to w are the matrix elements of the matrix

$$\sum_{\substack{z \in W \\ i \in \{1, \ldots, \mu(z,w)\} \\ j \in \{1, \ldots, \mu(y,z)\}}} a_{z,w}^i A_{z,w}^i \circ a_{y,z}^j A_{y,z}^j = 0,$$

where $\circ$ is the matrix multiplication.

5.2. Thus, all we need to get the relations are the matrices (5.1.1) spanning

$$\operatorname{Hom}_C^1(\mathbb{V}_y, \mathbb{V}_w).$$

Algorithmically, it is enough to find homogeneous bases in $\mathbb{V}_w$ for $w \in W$ with the explicit action of the generators

$$\sigma_{s_i}, \qquad i = \{1 \ldots r\}$$

of C. Then we can take all linear maps of degree 1 between $\mathbb{V}_y$ and $\mathbb{V}_w$ and solve a system of linear equations to find those linear maps which commute with the action of (the generators of) C.

5.3. We will find a homogeneous basis in $\mathbb{V}_w$ by induction on the length of the element $w \in W$.

Step 0. First of all, if $w = 1$, then $\mathbb{V}_1 = \mathbb{C}$.

Step 1. By induction, suppose that we already have homogeneous bases with the explicit action of (the generators of) C in $\mathbb{V}_y$ for $y < w$. Take a reduced decomposition $w = s_1 \ldots s_l$. It is easy to see from Theorem 3.2 that

$$\mathrm{Hom}_C^0(\mathbb{V}_y, \mathbb{V}_w) = \begin{cases} \mathbb{C}, & y = w \\ 0, & \text{otherwise} \end{cases}$$

From that and the decomposition 4.3.1 we can find matrices (with respect to the induction-assumed basis in $\mathbb{V}_y$ and the basis (4.4.1) in $C_{s_1 \ldots s_l}$) spanning the space

$$\mathrm{Hom}_C^0(\mathbb{V}_y, C_{s_1 \ldots s_l})$$

by considering degree 0 maps between $\mathbb{V}_y$ and $C_{s_1 \ldots s_l}$ and making sure they commute with the action of the generators of C by solving a system of linear equations. Applying these matrices to the induction-assumed basis in $\mathbb{V}_y$ we get a homogeneous linearly independent system of vectors $v_1, \ldots, v_m$ in $C_{s_1 \ldots s_l}$ spanning the subspace

$$Y = \bigoplus_{y < w} \mathbb{V}_y^{n(y)}.$$

Step 2. Now we have a C-module $U = C_{s_1 \ldots s_l}$ with the basis (4.4.1) which we will number in an arbitrary way as $x_1, \ldots, x_{2^l}$, and its C-submodule

$$Y = \bigoplus_{y < w} \mathbb{V}_y^{n(y)}$$

with the basis $v_1, \ldots, v_m$ which we know in terms of the basis $x_1, \ldots, x_{2^l}$:

$$v_i = \sum_k \alpha_i^k x_k.$$

We need to find a basis in the C-module U/Y with the explicit action of (the generators of) C. This is a problem of computational commutative algebra algorithmic solutions to which are well known, cf. e.g. [16, Chapter 15]. Let us just outline the algorithm:

- take a basis element x_i not in Y and generate a C-submodule $Cx_i \subset U$;
- record a basis in $Cx_i - Y$ consisting of (linear combinations of) the elements of the form $\sigma_w x_i$, $w \in W$
- if $Cx_i + Y \neq U$, take another basis element x_j not in $Cx_i + Y$, and generate a C-submodule Cx_j.
- proceed until $Cx_i + Y = U$. The algorithm will stop since $\dim_{\mathbb{C}} U = 2^l < \infty$.

This algorithm involves arbitrary choices of basis elements $x_i, x_j, \ldots$.

Step 3. Proceeding by induction we find a homogeneous basis in $\mathbb{V}_w$, for all $w \in W$. Due to the choices above this basis will depend on the particular realization of the algorithm.

5.4. **Remark.** In order to improve the performance of an actual computer realization of this algorithm, one should use a number of shortcuts. For example, instead of using the module $C_{s_1 \ldots s_l}$ to extract a basis in $\mathbb{V}_w$, we could use $C \otimes_{C^s} \mathbb{V}_{w'}$, where $w = w's$ and $l(w) = l(w') + 1$. (I am grateful to T. Braden for this and other suggestions.) A detailed implementation of the algorithm in the A_2 case is worked out in the Appendix (Section 8).

5.5. **Remark: one quiver = all quivers.** Recall the Koszul algebra A underlying the category $\mathcal{O}_0 \simeq \mathcal{P}_{\mathrm{Schubert}}(G/B)$ from [4] and 3.4. The algebra A is generated over A_0 by A_1 and it is quadratic, that is the ideal of relations is generated by an A_0-A_0-bimodule

$$R \subset A_1 \otimes_{A_0} A_1.$$

The quiver of the category $\mathcal{O}_0$ in this language is just a choice of basis in A_1 (arrows) and a choice of basis in the subspace R (relations between arrows). Once we have a basis in A_1 and a basis in R we can obtain any other bases in A_1 and R by linear transformations.

6. WEIGHT FILTRATION AND THE FUNCTOR $\mathcal{H}$

In this section we construct a functor $\mathcal{H}$ from mixed perverse sheaves to mixed IC-modules.

6.1. Let $\mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X)$ be a mixed category of mixed perverse sheaves. Examples of such categories include [4, 4.4, 4.5] and [40, 10]. Let $v : \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X) \to \mathcal{P}_{\mathcal{S}}(X)$ be the degrading functor.

By [4, Lemma 4.1.2], cf. [3], every object $A \in \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X)$ has a unique finite decreasing weight filtration $W_\bullet = W_\bullet A$ such that $W_m A / W_{m-1} A$ is a pure object of weight m.

6.1.1. Suppose that we have a functor $K : \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X) \to C^b(\mathrm{Vec})$ to the category of bounded complexes of (graded) R-modules such that for $A \in \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X)$ the cohomology of $K(A)$ is the hypercohomology of A and the weight filtration of A induces a decreasing filtration

$$(6.1.1) \qquad K(A) = K(A) \supseteq \cdots \supseteq K(W_m A) \supseteq K(W_{m-1} A) \cdots$$

on $K(A)$, and moreover, the cohomology of $K(W_m A)/K(W_{m-1} A)$ is the hypercohomology of $W_m A / W_{m-1} A$.

6.2.   Assuming the statement of The $d^2 = 0$ Theorem, one could take the total complex of the corresponding (non-mixed) IC-module corresponding to $v(A)$ as the complex $K(A)$ for $A \in \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X)$. In particular, we can do it for simplicial complexes and flag varieties. We will fix this choice of K for the rest of this section.

Consider now the first term $E^1 = E^1(A)$ of the spectral sequence associated to the filtration (6.1.1). One could look at E^1 as a mixed IC-module. Indeed, for $A \in \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X)$ the stalks of the corresponding IC-module $V = \mathcal{H}(A)$ are determined from the formula

$$H^i \left(K(W_m A)/K(W_{m-1} A) \right) = \bigoplus_S V_S^{-m} \otimes IH^i(\overline{S})$$

and the boundary maps of V are the differentials of E^1. The total complex of V constructed in this way is precisely the diagonal complex of E^1.

Thus we have constructed a functor $\mathcal{H} : \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X) \rightarrow \mathcal{A}_{\mathcal{S}}^{\mathrm{mixed}}(X)$. Observe that the spectral sequence degenerates at the second term and so the total complex of the IC-module $\mathcal{H}(A)$ calculates the hypercohomology of A for $A \in \mathcal{P}_{\mathcal{S}}^{\mathrm{mixed}}(X)$. In other words we have the following

Theorem. *Assuming the statement of* The $d^2 = 0$ Theorem, *the total complex of the mixed* IC-*module calculates the hypercohomology of the corresponding mixed perverse sheaf.*

6.3.   Let $\widetilde{\mathcal{P}}(G/B)$ be the mixed category of mixed perverse sheaves considered in [4, 4.4]. Let $\widetilde{\mathcal{A}}(G/B)$ be the subcategory of the category $\mathcal{A}^{\mathrm{mixed}}(G/B)$ with coefficients in $\mathbb{Q}_l$ consisting of objects V such that

$$V_S^i = 0 \qquad \text{unless} \qquad \dim S = i \mod 2.$$

Then the functor $\mathcal{H} : \widetilde{\mathcal{P}}(G/B) \rightarrow \widetilde{\mathcal{A}}(G/B)$ provides the mixed version of the equivalence established in Section 3.

7. Further directions and final remarks

7.1. **A combinatorial challenge.** The basis in $\mathbb{V}_w$ we constructed in 5.3 is very non-canonical: it depends on the reduced decompositions of w we choose for each w, and on the way we number our bases. However, once we have found *one* basis in $\mathbb{V}_w$ we can obtain *any other* basis by a (grading-preserving) linear transformation.

It would be very interesting to construct a *distinguished* homogeneous basis in the C-module $\mathbb{V}_w$ for all $w \in W$ independent of any choices. The basis in question should specialize to the basis of Schubert cycles when the variety $\overline{X}_w$ is smooth, and the formulas for the action of C should generalize the Chevalley's formula (4.1.1).

7.2. **Moment graphs.** In [12] T. Braden and R. MacPherson give another construction of a module structure on $IH^*(\overline{X}_w)$ and $IH_T^*(\overline{X}_w)$, where the latter is the T-equivariant intersection homology. One could use their techniques to compute with IC-modules. It would also be very interesting to define T-*equivariant* IC-modules, perhaps related to T-equivariant perverse sheaves and *singular* blocks of $\mathcal{O}$.

7.3. **Parabolic and singular case.** In Section 5 we have considered only the full flag variety and the regular block of the category $\mathcal{O}_0$. However, the algorithm goes through without change for a parabolic flag variety G/P and the corresponding regular parabolic block of the category $\mathcal{O}$. The parabolic Chevalley's formula is known, cf. [17]. The quiver for a singular block can be obtained by Koszul (and thus quadratic) duality from the corresponding parabolic quiver, cf [4].

7.4. **Harish-Chandra modules.** In fact, in [18, Section 1] I. M. Gelfand asks for a quiver describing (a block of) the category of Harish-Chandra modules over an arbitrary simple group. Harish-Chandra modules reduce to category $\mathcal{O}$ in some cases [8]. We hope that our methods will provide an answer to that question as well, and help prove Soergel's Langlands duality conjecture, cf. [37]. (There are even more general representation categories, related to arbitrary Coxeter systems, for which the theory partially goes through, cf. [15].)

7.5. **Toric varieties.** First we introduce the *relative* IC-modules. Let $D(X) = D_c^b(X)$ be the bounded derived category of constructible sheaves on X. We could define a relative version of IC-modules by using another ∂-functor $D_c^b(X) \to D^b(A)$, where A is an abelian category, instead of hypercohomology. A good example of such a functor would be the derived functor $Rf_* : D_c^b(X) \to D_c^b(Y)$ associated to a map $f : X \to Y$.

It would be interesting to look at the relative IC-modules related to the Koszul category of perverse sheaves on toric varieties studied in [10]. Let X be a toric variety with the action of the torus T and let $\mu : X \to X/T$ be the quotient map. The relative IC-modules with respect to μ would still have vector spaces V_σ as stalks at each face σ, and elements of the space

$$\mathrm{Hom}^1_{\mu_* R_X}(\mu_*\mathbf{IC}(\sigma, V_\sigma), \mu_*\mathbf{IC}(\tau, V_\tau))$$

for any two faces (T-orbits) σ and τ as boundary maps, satisfying the $d^2 = 0$ axiom. (Here μ_* is the functor between derived categories, and R_X is the constant sheaf on X.) The objects $\mu_* R_X$, $\mu_*\mathbf{IC}_\sigma$, and $\mu_*\mathbf{IC}_\tau$ have a very nice combinatorial description, cf. [10, 1.3].

7.6. **Affine Grassmannians.** The $d^2 = 0$ Theorem trivially holds for the semisimple category of $G(O)$-equivariant perverse sheaves of $\mathbb{C}$-vector spaces on the affine Grassmannian $G(K)/G(O)$. It would be interesting to prove an analogous result for a suitable Koszul category of perverse sheaves of vector spaces over a field of positive characteristic.

7.7. **Saper's L-modules.** In a remarkable recent paper [34] L. Saper introduced the notion of L-modules, in order to prove a conjecture of Rapoport and Goresky-MacPherson. It seems that L-modules should be related to IC-modules on the (compactifications of) locally symmetric spaces. It would be very interesting to understand the relationship between these structures and perverse sheaves.

7.8. **Generalized H-modules.** If we take the usual homology instead of intersection homology in the definition of IC-modules, we will get the usual triangulation-constructible sheaves on simplicial complexes, cf. [25, 29, 40]. Thus we have a machine transforming a homology theory into an abelian (or, more generally, A_∞) category of "sheaves." It would be intriguing to consider IC-modules with an arbitrary (generalized) homology theory H instead of intersection homology.

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8. Appendix. The quiver in the A_2 case

The quiver relations in the A_2 case were obtained in [23, 38], but we would still like to illustrate our quiver algorithm from section 5 on this example. The quiver we obtain is defined "over the integers", cf. 8.6.

8.1. **Cohomology ring.** In the A_2 case the Weyl group $W = \mathfrak{S}_3 = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$ and the cohomology ring $C = H^*(G/B)$ has a basis of 6 elements: $C = \mathrm{span}\{1, \sigma_{s_1}, \sigma_{s_2}, \sigma_{s_1 s_2}, \sigma_{s_2 s_1}, \sigma_{s_1 s_2 s_1}\}$. For simplicity we denote $\sigma_i = \sigma_{s_i}$, $i = 1, 2$.

Using the Chevalley's formula (4.1.1) we compute the following partial multiplication table for this algebra:

	σ_1	σ_2
1	σ_1	σ_2
σ_1	$\sigma_{s_2 s_1}$	$\sigma_{s_2 s_1} + \sigma_{s_1 s_2}$
σ_2	$\sigma_{s_2 s_1} + \sigma_{s_1 s_2}$	$\sigma_{s_1 s_2}$
$\sigma_{s_1 s_2}$	$\sigma_{s_1 s_2 s_1}$	0
$\sigma_{s_2 s_1}$	0	$\sigma_{s_1 s_2 s_1}$
$\sigma_{s_1 s_2 s_1}$	0	0

8.1.1. The subalgebras C^{s_i}, $i = 1, 2$ presented in subsection 4.2 can be explicitly described as follows:

$$C^{s_1} = \mathrm{span}\{1, \sigma_2, \sigma_{s_1 s_2}\},$$

and

$$C^{s_2} = \mathrm{span}\{1, \sigma_1, \sigma_{s_2 s_1}\}.$$

8.1.2. It is easy to see the decomposition (4.2.1) in this case:

$$\sigma_1 \cdot C^{s_1} \oplus C^{s_1} = \mathrm{span}\{\sigma_1, \sigma_{s_2 s_1} + \sigma_{s_1 s_2}, \sigma_{s_1 s_2 s_1}\} \oplus \mathrm{span}\{1, \sigma_2, \sigma_{s_1 s_2}\} = C,$$

and

$$\sigma_2 \cdot C^{s_2} \oplus C^{s_2} = \mathrm{span}\{\sigma_2, \sigma_{s_2 s_1} + \sigma_{s_1 s_2}, \sigma_{s_1 s_2 s_1}\} \oplus \mathrm{span}\{1, \sigma_1, \sigma_{s_2 s_1}\} = C.$$

8.2. **The modules $\mathbb{V}_w$.** These modules can be explicitly described as follows:

8.2.1. $\mathbb{V}_1 = \mathbb{C}$, and σ_i, $i = 1, 2$ act by 0.

8.2.2. $\mathbb{V}_{s_1} = C \otimes_{C^{s_1}} \mathbb{C} = \text{span}\{1 \otimes 1, \sigma_1 \otimes 1\}$. Action of σ_1:

$$\sigma_1 \cdot 1 \otimes 1 = \sigma_1 \otimes 1$$

and

$$\sigma_1 \cdot (\sigma_1 \otimes 1) = \sigma_1^2 \otimes 1 = \sigma_{s_2 s_1} \otimes 1 = (\sigma_1 \sigma_2 - \sigma_{s_1 s_2}) \otimes 1 = \sigma_1 \otimes \sigma_2 \cdot 1 - 1 \otimes \sigma_{s_1 s_2} \cdot 1 = 0.$$

Action of σ_2:

$$\sigma_2 \cdot 1 \otimes 1 = \sigma_2 \otimes 1 = 1 \otimes \sigma_2 \cdot 1 = 0$$

and

$$\sigma_2 \cdot (\sigma_1 \otimes 1) = \sigma_1 \sigma_2 \otimes 1 = \sigma_1 \otimes \sigma_2 \cdot 1 = 0.$$

Thus the action of σ_1 and σ_2 in this basis is given by the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

8.2.3. $\mathbb{V}_{s_2} = C \otimes_{C^{s_2}} \mathbb{C} = \text{span}\{1 \otimes 1, \sigma_2 \otimes 1\}$. The action of σ_1 and σ_2 in this basis is given by the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

8.2.4. $\mathbb{V}_{s_1 s_2} = C \otimes_{C^{s_2}} C \otimes_{C^{s_1}} \mathbb{C}$. Note that

$$\mathbb{V}_{s_1 s_2} = \text{span}\{1 \otimes 1 \otimes 1, \sigma_2 \otimes 1 \otimes 1, 1 \otimes \sigma_1 \otimes 1, \sigma_2 \otimes \sigma_1 \otimes 1\}.$$

The action of σ_1 and σ_2 in this basis is given by the matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

8.2.5. $\mathbb{V}_{s_2 s_1} = C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} \mathbb{C}$. Note that

$$\mathbb{V}_{s_1 s_2} = \text{span}\{1 \otimes 1 \otimes 1, \sigma_1 \otimes 1 \otimes 1, 1 \otimes \sigma_2 \otimes 1, \sigma_1 \otimes \sigma_2 \otimes 1\}.$$

The action of σ_1 and σ_2 in this basis is given by the matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

8.2.6. $C_{s_1 s_2 s_1} = C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} C \otimes_{C^{s_1}} \mathbb{C}$ We have

$$C_{s_1 s_2 s_1} = \text{span}\{1 \otimes 1 \otimes 1 \otimes 1, 1 \otimes 1 \otimes \sigma_1 \otimes 1,$$
$$1 \otimes \sigma_2 \otimes 1 \otimes 1, \sigma_1 \otimes 1 \otimes 1 \otimes 1,$$
$$1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1, \sigma_1 \otimes 1 \otimes \sigma_1 \otimes 1,$$
$$\sigma_1 \otimes \sigma_2 \otimes 1 \otimes 1, \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1\}$$

The action table is as follows:

	σ_1	σ_2
$1 \otimes 1 \otimes 1 \otimes 1$	$\sigma_1 \otimes 1 \otimes 1 \otimes 1$	$1 \otimes \sigma_2 \otimes 1 \otimes 1$
$1 \otimes 1 \otimes \sigma_1 \otimes 1$	$\sigma_1 \otimes 1 \otimes \sigma_1 \otimes 1$	$1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1$
$1 \otimes \sigma_2 \otimes 1 \otimes 1$	$\sigma_1 \otimes \sigma_2 \otimes 1 \otimes 1$	$1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1$
$\sigma_1 \otimes 1 \otimes 1 \otimes 1$	$\sigma_1 \otimes \sigma_2 \otimes 1 \otimes 1 - 1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1$	$\sigma_1 \otimes \sigma_2 \otimes 1 \otimes 1$
$1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1$	$\sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1$	0
$\sigma_1 \otimes 1 \otimes \sigma_1 \otimes 1$	$\sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1$	$\sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1$
$\sigma_1 \otimes \sigma_2 \otimes 1 \otimes 1$	$\sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1$	$\sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1$
$\sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1$	0	0

The decomposition (4.3.1) in this case is $C_{s_1 s_2 s_1} = \mathbb{V}_{s_1 s_2 s_1} \oplus \mathbb{V}_{s_1}$. The submodule isomorphic to $\mathbb{V}_{s_1}$ is spanned in $C_{s_1 s_2 s_1}$ by:

$$\mathbb{V}_{s_1} \hookrightarrow C_{s_1 s_2 s_1}$$
$$1 \otimes 1 \mapsto 1 \otimes 1 \otimes \sigma_1 \otimes 1 - 1 \otimes \sigma_2 \otimes 1 \otimes 1$$
$$\sigma_1 \otimes 1 \mapsto \sigma_1 \otimes 1 \otimes \sigma_1 \otimes 1 - \sigma_1 \otimes \sigma_2 \otimes 1 \otimes 1$$

Following Step 2 of section 5.3, choose a basis element $1 \otimes 1 \otimes 1 \otimes 1 \in C_{s_1 s_2 s_1}$ and generate a C-submodule $C \cdot (1 \otimes 1 \otimes 1 \otimes 1) \subset C_{s_1 s_2 s_1}$. We have

$$C \cdot (1 \otimes 1 \otimes 1 \otimes 1) = \text{span}\{1 \otimes 1 \otimes 1 \otimes 1,$$
$$\sigma_1 \otimes 1 \otimes 1 \otimes 1,$$
$$1 \otimes \sigma_2 \otimes 1 \otimes 1,$$
$$\sigma_1 \otimes \sigma_2 \otimes 1 \otimes 1 - 1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1,$$
$$1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1$$
$$\sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1\}$$

We see that $C \cdot (1 \otimes 1 \otimes 1 \otimes 1) + \mathbb{V}_{s_1} = C_{s_1 s_2 s_1}$, so $\mathbb{V}_{s_1 s_2 s_1} = C \cdot (1 \otimes 1 \otimes 1 \otimes 1)$ with the basis as above. The matrices of σ_1 and σ_2 in this basis are as

follows:

$$\sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

8.2.7. Notice that for $\mathbb{V}_{s_1 s_2 s_1} = C$ as a C-module, so we could also use the basis of "Schubert cycles" $\{1, \sigma_{s_1}, \sigma_{s_2}, \sigma_{s_1 s_2}, \sigma_{s_2 s_1}, \sigma_{s_1 s_2 s_1}\}$.

8.3. Matrices representing Hom^1.

8.3.1. *Setup.* For any pair $w, y \in W = \mathfrak{S}_3$. We need to find matrices representing homomorphisms $f_{w,y} : \mathbb{V}_w \to \mathbb{V}_y$ of degree 1 such that $\sigma_i f = f \sigma_i$ for $i = 1, 2$. We use the bases in $\mathbb{V}_w$ constructed the section 8.2.

We will collect these matrices into one matrix of the map $d : \oplus_w \mathbb{V}_w \to \oplus_w \mathbb{V}_w$. Since $\dim \oplus_w \mathbb{V}_w = 19$, the matrix of d has dimension 19×19, and it has 16 nonzero blocks of the form $f_{w,y}$, $w, y, \in \mathfrak{S}_3$.

(8.3.1)
$$d = \begin{pmatrix} 0 & f_{s_1,1} & f_{s_2,1} & 0 & 0 & 0 \\ f_{1,s_1} & 0 & 0 & f_{s_1 s_2, s_1} & f_{s_2 s_1, s_1} & 0 \\ f_{1,s_2} & 0 & 0 & f_{s_1 s_2, s_2} & f_{s_2 s_1, s_2} & 0 \\ 0 & f_{s_1, s_1 s_2} & f_{s_2, s_1 s_2} & 0 & 0 & f_{s_1 s_2 s_1, s_1 s_2} \\ 0 & f_{s_1, s_2 s_1} & f_{s_2, s_2 s_1} & 0 & 0 & f_{s_1 s_2 s_1, s_2 s_1} \\ 0 & 0 & 0 & f_{s_1 s_2, s_1 s_2 s_1} & f_{s_2 s_1, s_1 s_2 s_1} & 0 \end{pmatrix}.$$

8.3.2. *The matrices $f_{w,y}$, $w, y, \in \mathfrak{S}_3$.* We record the 16 matrices $f_{w,y}$ obtained by solving the linear systems $\sigma_i f = f \sigma_i$ for $i = 1, 2$.

$$f_{1,s_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad f_{1,s_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad f_{s_1,1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad f_{s_2,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$f_{s_1, s_1 s_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad f_{s_1, s_2 s_1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f_{s_2, s_1 s_2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad f_{s_2, s_2 s_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f_{s_1s_2,s_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad f_{s_2s_1,s_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$f_{s_1s_2,s_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad f_{s_2s_1,s_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$f_{s_1s_2,s_1s_2s_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad f_{s_2s_1,s_1s_2s_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$f_{s_1s_2s_1,s_1s_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad f_{s_1s_2s_1,s_2s_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

8.4. The quiver notation.

8.4.1. *Vertex enumeration.* The quiver of the algebra in question looks as follows (here we actually have two arrows in the opposite directions between every two vertices connected by the graphical image $\leftrightarrow$)

(8.4.1)
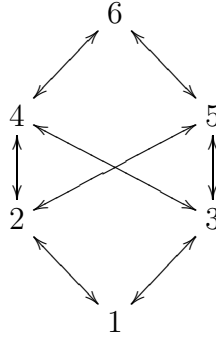
where in order to harmonize the notation with [23, 38] we number the elements of $\mathfrak{S}_3$ as follows:

(1) $s_1s_2s_1$ is number 1,
(2) s_1s_2 is number 2,
(3) s_2s_1 is number 3,
(4) s_1 is number 4,
(5) s_2 is number 5,
(6) 1 is number 6.

8.4.2. *Arrow notation.* We have 16 arrows on the quiver. The arrows will be denoted by specifying the numbers of their initial and terminal vertices. For example the arrow from s_1 to 1 (vertex number 4 to vertex number 6) will be denoted as (46).

8.4.3. *Path notation.* A path on the quiver will be denoted by the sequence of the vertices through which it goes. For example the path of length 2 starting at $s_1 s_2 s_1$ (vertex number 1) going through $s_1 s_2$ (vertex number 2) and ending at s_1 (vertex number 4) will be denoted by the sequence (124).

8.4.4. *The free quiver algebra.* Consider the algebra freely generated by the arrows of the quiver above, i.e. the tensor algebra $T(A_1) = k \oplus \bigoplus_{i=1}^{\infty} A_1^{\otimes i}$ where k is the semisimple ring spanned by the idempotents at the vertices and A_1 is the $k - k$-bimodule spanned by the arrows of the quiver, cf. 3.5.

8.5. **The quiver relations.** First let us relabel the matrices $f_{y,w}$ using the enumeration of $y \in \mathfrak{S}_3$ as above. Thus the matrix $f_{s_1 s_2 s_1, s_2 s_1}$ will now be denoted as $f_{(13)}$. Introduce the matrix $\widetilde{d}$ with the coefficients in $T(A_1)$ given by
(8.5.1)
$$\widetilde{d} = \begin{pmatrix} 0 & (46)f_{(46)} & (56)f_{(56)} & 0 & 0 & 0 \\ (64)f_{(64)} & 0 & 0 & (24)f_{(24)} & (34)f_{(34)} & 0 \\ (65)f_{(65)} & 0 & 0 & (25)f_{(25)} & (35)f_{(35)} & 0 \\ 0 & (42)f_{(42)} & (52)f_{(52)} & 0 & 0 & (12)f_{(12)} \\ 0 & (43)f_{(43)} & (53)f_{(53)} & 0 & 0 & (13)f_{(13)} \\ 0 & 0 & 0 & (21)f_{(21)} & (31)f_{(31)} & 0 \end{pmatrix}.$$

For example,
$$(42)f_{(42)} == \begin{pmatrix} 0 & 0 \\ (42) & 0 \\ -(42) & 0 \\ 0 & (42) \end{pmatrix}.$$

Some more notation: denote $f_{(ijk)} = f_{jk} \circ f_{ij}$, where $\circ$ stands for matrix multiplication.

8.5.1.   Now, consider the 19×19 matrix $\widetilde{d^2}$ as a matrix over $T(A_1)$.

$$\widetilde{d^2} = \begin{pmatrix} g_{66} & 0 & 0 & g_{26} & g_{36} & 0 \\ 0 & g_{44} & g_{54} & 0 & 0 & g_{14} \\ 0 & g_{45} & g_{55} & 0 & 0 & g_{15} \\ g_{62} & 0 & 0 & g_{22} & g_{32} & 0 \\ g_{63} & 0 & 0 & g_{23} & g_{33} & 0 \\ 0 & g_{41} & g_{51} & 0 & 0 & g_{11} \end{pmatrix}.$$

For example,

$$g_{23} = (243)f_{(243)} + (253)f_{(253)} + (213)f_{(213)}$$

$$(8.5.2) \qquad = \begin{pmatrix} 0 & 0 & 0 & 0 \\ (253) + (213) & 0 & 0 & 0 \\ (243) - (253) & 0 & 0 & 0 \\ 0 & (253) + (213) & (243) + (213) & 0 \end{pmatrix}.$$

In general,

$$g_{66} = (646)f_{(646)} + (656)f_{(656)}$$
$$g_{26} = (246)f_{(246)} + (256)f_{(256)}$$
$$g_{36} = (346)f_{(346)} + (356)f_{(356)}$$
$$g_{44} = (464)f_{(464)} + (424)f_{(424)} + (434)f_{(434)}$$
$$g_{54} = (564)f_{(564)} + (524)f_{(524)} + (534)f_{(534)}$$
$$g_{45} = (465)f_{(465)} + (425)f_{(425)} + (435)f_{(435)}$$
$$g_{55} = (565)f_{(565)} + (525)f_{(525)} + (535)f_{(535)}$$
$$g_{14} = (124)f_{(124)} + (134)f_{(134)}$$
$$g_{15} = (125)f_{(125)} + (135)f_{(135)}$$
$$g_{62} = (642)f_{(642)} + (652)f_{(652)}$$
$$g_{63} = (643)f_{(643)} + (653)f_{(653)}$$
$$g_{22} = (242)f_{(242)} + (252)f_{(252)} + (212)f_{(212)}$$
$$g_{32} = (342)f_{(342)} + (352)f_{(352)} + (312)f_{(312)}$$
$$g_{23} = (243)f_{(243)} + (253)f_{(253)} + (213)f_{(213)}$$
$$g_{33} = (343)f_{(343)} + (353)f_{(353)} + (313)f_{(313)}$$
$$g_{41} = (421)f_{(421)} + (431)f_{(431)}$$
$$g_{51} = (521)f_{(521)} + (531)f_{(531)}$$
$$g_{11} = (121)f_{(121)} + (131)f_{(131)}.$$

Each of the matrices g_{ij} above can be presented in the explicit form (8.5.2).

8.5.2. *A basis of relators.* Thus $\widetilde{d^2}$ is a matrix with elements in $A_1 \otimes_k A_1$. By section 5.1 the elements of $\widetilde{d^2}$ span the submodule of relators $R \subset A_1 \otimes_k A_1$. One can choose a linearly independent subsystem (basis of R) from elements of $\widetilde{d^2}$ for example as follows:

$$
\begin{aligned}
\{&(121), (131), \\
&(242), (252) + (212), \\
&(353), (343) + (313), \\
&(243) + (213), (253) + (213), \\
&(352) + (312), (342) + (312), \\
&(124) + (134), (125) + (135), \\
&(246) + (256), (346) + (356), \\
&(421) + (431), (521) + (531), \\
&(464) - (424), (565) - (535), \\
&(425) + (435) + (465), (524) + (534) + (564), \\
&(642) + (652), (643) + (653)\}.
\end{aligned}
$$

(8.5.3)

There are 22 relators as $\dim R = 22$.

8.6. **Open question.** Notice that all the coefficients of basic paths of lengths two in (8.5.3) are ± 1 as opposed to [38] where the coefficients are more general integers. It would be interesting to know whether coefficients of basic paths of lengths two for the relators can be chosen to be ± 1 in arbitrary type.

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